Inversion of First-Order Perturbation Theory and Its Application to Structural Design

Karl A. Stetson* and Gary E. Palma†
United Technologies Research Center, East Hartford, Conn.

Present methods of structural design, such as the finite-element method, NASTRAN, are capable of accurately determining the shape and frequencies of the normal modes of complicated structures, if the stiffness and mass matrices are specified. The inverse problem, that of determining the stiffness and mass matrices of a structure in order to arrive at a desired set of normal mode shapes and frequencies, in general is unsolved. A mathematical method is presented here which addresses the inverse problem by the use of first-order perturbation theory. In this approach, the usual perturbation calculation is inverted. That is, the structural changes necessary to effect a given change in vibration modes are computed, rather than computing the effect on the vibration modes resulting from a small change in the design of that structure. The application of this technique requires a specification of a number of the normal modes of vibration of a base line structural design. This information can be obtained by a conventional finite-element calculation or by measurement of the normal modes of an existing prototype. In the former case, the inverse perturbation calculation can be used to proceed from the baseline design to an optimum design by an iterative procedure. In the latter case, the inverse perturbation calculation employs normal mode data from an existing structure to determine the changes in that structure necessary to improve its vibratory response. The basic mathematical structure of the inverse perturbation technique is presented herein, together with two illustrative numerical examples, the first employing four springs and three masses, and the second treating the modification of the modes of a cantilever beam.

Introduction

IN a previous paper,[1] a mathematical treatment was presented that applied first-order perturbation theory to the prediction of changes in mode shapes and frequencies of undamped structures, resulting from small changes in mass or stiffness. Although application of the previous work can offer insight into the consequences of changing the design of a structure, it leaves unanswered the question of exactly what combination of structural changes should be made to accomplish a prescribed set of changes to the vibration modes of the structure. The problem becomes particularly difficult when a significant number of simultaneous changes in mode shapes and frequencies are desired. Design methods that employ trial and error easily can miss configurations that might accomplish the desired goal and lead to the false conclusion that no satisfactory design exists.

There have been several efforts in recent years to use the results of vibration tests to correct or modify a mathematical model of a structure.[2,3] Although the methods presented herein can be applied to the correlation and modification of structural dynamic models, the major application is expected to be in the area of design synthesis. In particular, this paper presents an inversion of the previous perturbation theory of Ref. 1, whereby it becomes possible to determine a set of structural changes that will effect a set of prescribed changes in the mode shapes and frequencies of a structure. The formulation presented has the desirable feature that the structural changes that are computed are uniquely determined by the particular set of constraints imposed upon the mode changes. In general, it is neither necessary nor possible to specify all of the new modes of a structure, but rather it is possible to specify those new modes that are of engineering importance, and to design the structure so as to have those modes. If this is done, the additional modes of the structure simply change as a consequence of the new design in a way that is unspecified a priori, but which does not affect the modes which have been constrained.

Received June 20, 1975; revision received Oct. 1, 1975.
Index category: Structural Dynamic Analysis.

*Senior Research Engineer, Instrumentation Laboratory.
†Research Engineer, Instrumentation Laboratory.

This paper will begin by recalling the equations from the previous work which are essential to the inversion method for continuous structures. After presenting the inversion scheme, the method will be extended to a general lumped-parameter system of the type commonly used in finite-element numerical analyses. Following this theoretical treatment, two illustrative examples will be presented, the first employing four springs and three masses, and the second treating the modification of the modes of a cantilever beam.

Theory

Review of First-Order Perturbation

The previous discussion was restricted to a continuous system which could be described by a distributed set of stiffness moduli $K_q(x)$ and a distributed mass density $m(x)$, where x denotes three-dimensional space. The unperturbed normal modes $\phi_n(x)$ and mode frequencies ω_n satisfy the following energy equation:

$$K_n = \omega_n^2 M_n \tag{1}$$

where

$$K_n = \int \sum_q K_q(x) D_q\{\Phi_n(x)\}\mathrm{d}x = \text{modal stiffness} \tag{2}$$

$$M_n = \int m(x)\Phi_n^2(x)\,\mathrm{d}x = \text{modal mass} \tag{3}$$

with the orthogonality conditions

$$\int m(x)\Phi_n(x)\Phi_k(x)\,\mathrm{d}x = 0,\ n \neq k \tag{4}$$

The quantity $D_q\{\Phi_n(x)\}$ is a generalized nonlinear differential operator operating on $\Phi_n(x)$. For example, for a beam, $D_q\{\Phi_n(x)\}$ has the following form:

$$D_q\{\Phi_n(x)\}|_{\text{beam}} = (\mathrm{d}^2\Phi_n/\mathrm{d}x^2) \tag{5}$$

In perturbation theory, the mode shapes of the perturbed structure are assumed to be linear combinations of the mode shapes of the unperturbed structure, and the mode frequencies are slightly different from those of the unperturbed structure

$$\omega_n' = \omega_n + \Delta\omega_n \tag{6}$$

$$\Phi_n'(x) = \Phi_n(x) + \Delta\Phi_n(x) = \Phi_n(x) + \sum_{n\neq k} C_{nk}\Phi_k(x) \tag{7}$$

The primes denote the new structure which results from small changes $\Delta K_q(x)$ and $\Delta m(x)$ in stiffness and mass density, respectively. C_{nk} is the admixture coefficient that describes the amount of the kth mode that is subsumed by the nth mode in the primed system. Equations (1-7) can be used to derive equations relating the admixture coefficients and frequency changes to volume integrals of the stiffness and mass density changes. The result, to first order, can be expressed as the following set of equations, as discussed in Ref. 1:‡

$$\frac{\Delta\omega_n}{\omega_n} = \frac{1}{2}\,\frac{\int[\Sigma_q \Delta K_q(x) D_q\{\Phi_n\} - \omega_n^2 \Delta m(x)\Phi_n^2(x)]\,\mathrm{d}x}{\omega_n^2 M_n} \tag{8}$$

$$C_{kn}M_n + C_{nk}M_k = -\int \Delta m(x)\Phi_n(x)\Phi_k(x)\,\mathrm{d}x, \quad n\neq k \tag{9}$$

$$C_{kn}K_n + C_{nk}K_k = -\tfrac{1}{2}\int\Sigma_q \Delta K_q(x) D_{1q}\{\Phi_n,\Phi_k\}\,\mathrm{d}x, \; n\neq k \tag{10}$$

where $D_{1q}\{\Phi_n, \Phi_k\}$ is a symmetrical joint-differential operator that is obtained from $D_q\{\phi_n\}$ by a procedure described in Ref. 1. Equations (8-10) are in a form that is useful for calculating the $\Delta\omega_n$ and the C_{nk} when the mass and stiffness changes are prescribed and the original modes and frequencies are known.

Inversion of First-Order Perturbation Equations: Continuous System

An interesting problem arises when a specific change in the mode shapes and frequencies of a structure is required, and the necessary changes in the stiffness and mass are to be determined. This problem can be treated by rearranging Eqs. (8-10) in the form of a single matrix integral equation for the unknown functions $\Delta m(x)$ and $\Delta K_q(x)$. If Eq. (9) is multiplied by ω_n^2 and subtracted from Eq. (10), the admixture coefficients C_{kn} can be eliminated by making use of the identity $K_n = \omega_n^2 M_n$. The two resulting equations are

$$\left(\frac{\Delta\omega_n}{\omega_n}\right) = \frac{1}{2}\,\frac{[\Sigma_q \Delta K_q(x) D_q\{\Phi_n(x)\} - \omega_n^2 \Delta m(x)\Phi_n^2(x)]\,\mathrm{d}x}{\omega_n^2 M_n} \tag{11}$$

$$\left(\frac{\omega_n^2-\omega_k^2}{2\omega_n^2}\right)C_{nk} = \frac{1}{2}\,\frac{[\tfrac{1}{2}\Sigma_q \Delta K_q(x) D_{1q}\{\Phi_n,\Phi_k\} - \omega_n^2 \Delta m(x)\Phi_n\Phi_k]\,dx}{\omega_n^2 M_k}, n\neq k \tag{12}$$

At this point it is instructive to consider a one-dimensional isotropic system such as a beam of length ℓ, and to associate the changes in mass density and stiffness with a change in the thickness h of the beam. For this case, there is a single stiffness $K = EI$, and

$$\Sigma_q K_q D_q\{\Phi_n\} = EI(\Phi_n'')^2 \tag{13}$$

‡It should be noted that Eq. (8) includes a further approximation in which the change in the square of the mode frequencies has been linearized, i.e., $\Delta(\omega_n^2/\omega_n^2 \simeq 2\Delta\omega_n/\omega_n$. This approximation is accurate to 5% for a change in frequency of $\Delta\omega_{n/\omega_n} = 0.10$.

where EI is the flexural rigidity proportional to h^3, and $m(x)$ is mass per unit length proportionto to h.

The expressions for ΔK and Δm to first order in Δh then are given by

$$\Delta K = \Delta(EI) = 3K(\Delta h/h) \tag{14}$$

$$\Delta m = m(\Delta h/h) \tag{15}$$

Also, when $D\{\Phi_n\} = [\Phi_n''\}^2$, the bilinear joint-differential operators have the following form, as discussed in Ref. 1:

$$D_1\{\Phi_n, \Phi_k\} = 2\Phi_n''\Phi_k'' \tag{16}$$

If Eqs. (13-16) are substituted into Eqs. (11) and (12), the result can be written as a single matrix equation

$$\Delta_{nk} = \frac{1}{2}\left\{\int_0^\ell [3EI\Phi_n''\Phi_k'' - m\omega_n^2\Phi_n\Phi_k]\,\frac{\Delta h(x)}{h}\,\mathrm{d}x\right\}\Big/\omega_n^2 M_k \tag{17}$$

where the elements of the matrix Δ_{nk} have been defined as

$$\Delta_{nk} = \begin{cases} (\Delta\omega_n/\omega_n) & \text{for } n=k \\ [(\omega_n^2-\omega_k^2)]/2\omega_n^2]C_{nk} & \text{for } n\neq k \end{cases} \tag{18}$$

It should be noted that it is possible to put Eqs. (11) and (12) into the form of Eq. (17) directly from relationships presented in Ref. 1. The equations for the thin beam are used here mainly for simplicity and clarity. In its present form, Eq. (17) represents an infinite set of integral equations involving the unknown function $\Delta h(x)$. The general solution of Eq. (17) for $\Delta h(x)$ requires determination of the infinite set of modes of the unperturbed system, as well as a specification of the frequency changes and admixture coefficients for the entire set of perturbed modes. In practice, however, only a finite number of modes are known accurately, and the problem usually reduces to modifying some of these modes by making a prescribed change $\Delta h(x)$ and ignoring modes that are of no concern. In this case, Eq. (17) can be inverted by restricting $\Delta h(x)$ to be a sum of N prescribed functions, i.e., letting

$$\frac{\Delta h(x)}{h} = \sum_{p=1}^{N} A_p f_p(x) \tag{19}$$

and determining the N coefficients A_p. If Eq. (19) is substituted into Eq. (17), a set of simultaneous linear equations is obtained

$$\sum_{p=1}^{N} B_{(nk)p} A_p = \Delta_{nk} \tag{20}$$

where the coefficients $B_{(nk)p}$ are calculated from

$$B_{(nk)p} = \left\{\int_0^\ell [3\mathrm{EI}\Phi_n''\Phi_k'' \quad m\omega_n^2\Phi_n\Phi_k] f_p(x)\,\mathrm{d}x\right\}/\omega_n^2 M_k \tag{21}$$

The utility of Eq. (20) lies in the fact that there are just N unknowns in the infinite set of linear equations; i.e., the equations are overdetermined. We now may specify N constraints Δ_{nk} and reduce the infinite set of equations to an $N\times N$ set and solve for the coefficients A_p. Choice of these constraints restricts the calculation of the coefficients $B_{(nk)p}$ to only the N pairs of modes specified by the Δ_{nk}, and the modes not included in these pairs are not considered.

The number of independent constraints which can be effected is limited by either the number of accurately known modes or the number of functions employed in the representation of $\Delta h(x)$ [Eq. (19)]. In a particular problem, the Δ_{nk} are chosen to effect desired mode changes in terms of frequency changes and mode admixture. Thus, the changes that can be made are limited a priori by the number of modes known, the accuracy with which they are determined, and the ease of relative admixture. The functions $f_p(x)$ also should be chosen judiciously in order to simplify the integration of Eq. (21) and also to guarantee a nonsingular matrix $B_{(nk)p}$.

Inversion of First-Order Perturbation Equations: Finite-Element System

Equations (19-21) represent the formal solution of the inverse-perturbation problem, to first order, for a distributed one-dimensional system such as a beam. These results can be extended in a straightforward fashion to include two-dimensional structures such as flat plates and three-dimensional structures such as shells. However, the vibrational analysis of shells may be accomplished more readily by the application of finite-element numerical analysis. In this technique, the distributed structure is divided into a finite number of elements, and the modes of the structure are described by the motions of the individual elements. The extension of the equations of inverse perturbation to a finite-element formalism would permit the use of the latter in the design or modification of those structures that are analyzed best with finite-element models. It is thus important to develop the first-order inverse-perturbation equations for a finite-element system before proceeding to the solution of particular problems.

The equations for an undamped finite-element structure can be written in the concise matrix form[4]

$$\underset{\sim}{K} = \underset{\sim}{M}\underset{\sim}{\omega}^2 \tag{22}$$

the modal stiffness matrix $\underset{\sim}{K}$ and the modal mass matrix $\underset{\sim}{M}$ are defined by the following matrix equations:

$$\underset{\sim}{K} = \underset{\sim}{\Phi}^T \underset{\sim}{k}\, \underset{\sim}{\Phi} \tag{23}$$

$$\underset{\sim}{M} = \underset{\sim}{\Phi}^T \underset{\sim}{m}\, \underset{\sim}{\Phi} \tag{24}$$

where

$\underset{\sim}{k}$ = finite-element symmetric stiffness matrix
$\underset{\sim}{m}$ = finite-element diagonal mass matrix
$\underset{\sim}{\Phi}$ = finite-element mode matrix; Φ_{ni} is the ith component of the nth mode
$\underset{\sim}{\omega}^2$ = diagonal mode frequency matrix

Equations (22-24) are $N \times N$ matrix equations, where N is the number of degrees of freedom. The modes are orthogonal with respect to the mass matrix $\underset{\sim}{m}$ so that the modal mass matrix is diagonal. Since $\underset{\sim}{\omega}^2$ is diagonal, the modal stiffness matrix is also diagonal by Eq. (22).

As in the treatment of the distributed system, the perturbed mode shapes are written as linear combinations of the original mode shapes, with small changes in mode frequency according to Eqs. (6) and (7). These changes are implemented by making small changes in the stiffness and mass matrices. The perturbations can be written in matrix notation as

$$\underset{\sim}{\Phi}' = \underset{\sim}{\Phi} + \Delta\underset{\sim}{\Phi} = (\underset{\sim}{I} + \underset{\sim}{C})\underset{\sim}{\Phi} \tag{25}$$

$$\underset{\sim}{\omega}'^2 = \underset{\sim}{\omega}^2 + \Delta\underset{\sim}{\omega}^2 \tag{26}$$

$$\underset{\sim}{k}' = \underset{\sim}{k} + \Delta\underset{\sim}{k} \tag{27}$$

$$\underset{\sim}{m}' = \underset{\sim}{m} + \Delta\underset{\sim}{m} \tag{28}$$

where $\underset{\sim}{C}$ is a matrix whose elements C_{nk} are the admixture coefficients; note that $C_{nn} = 0$. Equations (22-24) can be written in the primed system as follows:

$$\underset{\sim}{K}' = \underset{\sim}{K} + \Delta\underset{\sim}{K} = (\underset{\sim}{M} + \Delta\underset{\sim}{M})\underset{\sim}{\omega}^2 + \underset{\sim}{M}\Delta\underset{\sim}{\omega}^2 \tag{29}$$

$$\underset{\sim}{K} + \Delta\underset{\sim}{K} = \underset{\sim}{K} + \underset{\sim}{C}\,\underset{\sim}{K} + \underset{\sim}{K}\,\underset{\sim}{C}^T + \underset{\sim}{\Phi}^T\Delta\underset{\sim}{k}\,\underset{\sim}{\Phi} \tag{30}$$

$$\underset{\sim}{M} + \Delta\underset{\sim}{M} = \underset{\sim}{M} + \underset{\sim}{C}\,\underset{\sim}{M} + \underset{\sim}{M}\,\underset{\sim}{C}^T + \underset{\sim}{\Phi}^T\Delta\underset{\sim}{m}\,\underset{\sim}{\Phi} \tag{31}$$

where terms of second order or higher in the perturbations have been neglected. Equations (30) and (31) are substituted into Eq. (29), and use is made of Eq. (22) to obtain a single matrix equation

$$\underset{\sim}{\Phi}^T\Delta\underset{\sim}{k}\,\underset{\sim}{\Phi} - \underset{\sim}{\omega}^2\underset{\sim}{\Phi}\,\Delta\underset{\sim}{m}\,\underset{\sim}{\Phi}^T = (\underset{\sim}{C}\,\underset{\sim}{\omega}^2 - \underset{\sim}{\omega}^2\underset{\sim}{C})\underset{\sim}{M} + \Delta\underset{\sim}{\omega}^2\underset{\sim}{M} \tag{32}$$

Equation (32) can be rewritten in terms of components to show the close similarity with Eqs. (11) and (12) for the continuous case. The result, after some rearrangement, can be written as

$$\Delta_{nk} = \tfrac{1}{2}\left(\Sigma_{i,j}\Phi_{ni}\Phi_{kj}\Delta k_{ij} - \omega_n^2\Sigma_i\Phi_{ni}\Phi_{ki}\Delta m_i\right)/\omega_n^2 M_k \tag{33}$$

where elements of Δ_{nk} are given in Eq. (18). A comparison of Eq. (33) with the results for the continuous case [Eqs. (11) and (12)] indicates that the finite-element case is obtained from the continuous case by replacing integrals by discrete sums, and by replacing the products of the stiffness moduli and the differential operators by the stiffness matrix elements. A more subtle difference arises, however, when the inversion of Eq. (33) is considered. The number of possible structural perturbations (i.e., the number of independent matrix elements Δk_{ij} and Δm_i) is, in general, less than the number of possible design constraints Δ_{nk}. For example, consider a linear spring-mass system composed of N masses and $N+1$ springs. There are $2N+1$ possible structural changes that can be implemented, but there are N^2 possible constraints for the mode shapes and frequencies. For $N>2$ the problem is overdetermined, even if all N modes are known accurately so that all of the constraints cannot be specified independently.

To include these restrictions in Eq. (33), the matrix elements Δk_i and Δm_i should be expressed in terms of the changes in stiffness and mass of the independent structural elements. If there are P independent stiffnesses k_p', and Q independent masses m_q', the following linear transformations can be used:

$$\Delta k_{ij} = \sum_{p=1}^{P} a_{ip}a_{jp}\Delta k_p' \tag{34}$$

$$\Delta m_i = \sum_{q=1}^{Q} b_{iq}\Delta m_q' \tag{35}$$

where a_{ip} and a_{jp} are elements of an $N \times P$ transformation matrix $\underset{\sim}{a}$ that relates changes in the elements of the symmetric stiffness matrix Δk_{ij} to changes in the independent stiffnesses $\Delta k_p'$. Similarly, b_{iq} is an $N \times Q$ matrix that relates changes in the diagonal mass matrix Δm_i to changes in the masses of the independent structural elements $\Delta m_q'$.

If Eqs. (34) and (35) are substituted into Eq. (33), a set of simultaneous linear equations in $(P+Q)$ unknowns results

$$\Delta_{nk} = \sum_{p=1}^{P} B^1_{(nk)p}\Delta k_p' + \sum_{q=1}^{Q} B^2_{(nk)q}\Delta m_q' \tag{36}$$

where

$$B^1_{(nk)p} = \frac{1}{2}\left(\sum_{i,j}^{N}\Phi_{ni}\Phi_{kj}a_{ip}a_{jp}/\omega_n^2 M_k\right) \tag{37}$$

$$B^2_{(nk)q} = -\frac{1}{2}\sum_{i=1}\frac{\Phi_{ni}\Phi_{ki}b_{iq}}{M_k} \tag{38}$$

As in the solution of the inversion problem for the distributed structure, the determination of the mass and stiffness changes for the finite-element system is obtained by matrix inversion.§

§It should be noted that the pair of indices (nk) in Eqs. (36-38) can be replaced by a single index simply by relabeling the various mode pairs in a consistant fashion.

There are $(P+Q)$ unknowns in the system of Eq. (36). It is possible (and necessary) only to specify $(P+Q)$ independent constraints Δ_{nk}, corresponding to P + Q of the available pairs of modes $\Phi_n \Phi_k$. Of course, the determining factor in an actual design problem may be the number of constraints that are needed to obtain a satisfactory design; this number usually will be much less than the number of independent mass and stiffness elements available. The importance of the inverse perturbation technique lies in the small number of modes that are required for the calculations when only a small number of design changes are to be made. In fact, the number of modes required is always less than, or equal to, one plus the number of constraints imposed. Note also that the mass and stiffness changes in general can be determined independently.

In many problems, the changes in mass and stiffness are not independent, as we have seen in the treatment of the beam equations using the continuum formalism. This can be included in the finite-element formalism in a straightforward fashion by imposing a linear relation between the elements $\Delta k_p'$ and $\Delta m_q'$ of Eqs. (34) and (35). This approach is also a useful technique for reducing the number of independent mass and stiffness elements when the latter greatly exceeds the required number of design changes.

It should be noted that there are additional restrictions on the choice of constraints that are related to a class of inadmissible constraints. Constraints of this type are readily identifiable, as they result in singular matrices in the inversion Eq. (20) or (36). As an example of an inadmissible constraint, suppose that we wish to specify Δ_{nk} and Δ_{kn} independently, but we restrict the structural changes to either changes in mass or stiffness only. Examination of Eqs. (36-38) indicates that Δ_{nk} and Δ_{kn} are not independent for $\Delta k_p' = 0$. The matrix $B^2{}_{(nk)q}$ of Eq. (38) is singular in this case, since the elements of two of its rows are proportional.

Illustrative Examples

Finite-Element Example: Linear Spring-Mass System

At this point, the necessary equations are available for the solution of some illustrative problems. The first example is a simple finite-element structure consisting of a linear spring-mass system. Consider the modes of a system consisting of three masses and four springs, constrained to move in a linear fashion as shown in Fig. 1a. For convenience, the unperturbed system is assumed to have equal masses m, and equal spring constants k.

The mode frequencies are obtained by setting the secular determinant equal to zero

$$\begin{vmatrix} 2k - m_1\omega^2 & -k & 0 \\ -k & 2k - m_2\omega^2 & -k \\ 0 & -k & 2k - m_3\omega^2 \end{vmatrix} = 0 \tag{39}$$

In the unperturbed system, $m_1 = m_2 = m_3 = m$, and Eq. (39) reduces to

$$(2k - m\omega^2)\,[\,(2k - m\omega^2)^2 - 2k^2] = 0 \tag{40}$$

Equation (40) has three roots, corresponding to the vibration frequencies of the three modes; they are, in order of increasing radian frequency

$$\omega_1 = (2k/m)^{1/2}\,(1 - 1/\sqrt{2})^{1/2} \tag{41a}$$

$$\omega_2 = (2k/m)^{1/2} \tag{41b}$$

$$\omega_3 = (2k/m)^{1/2}\,(1 + 1/\sqrt{2})^{1/2} \tag{41c}$$

The mode shapes ϕ_1, ϕ_2, and ϕ_3 can be obtained by substituting the mode frequencies of Eq. (41) into the equations

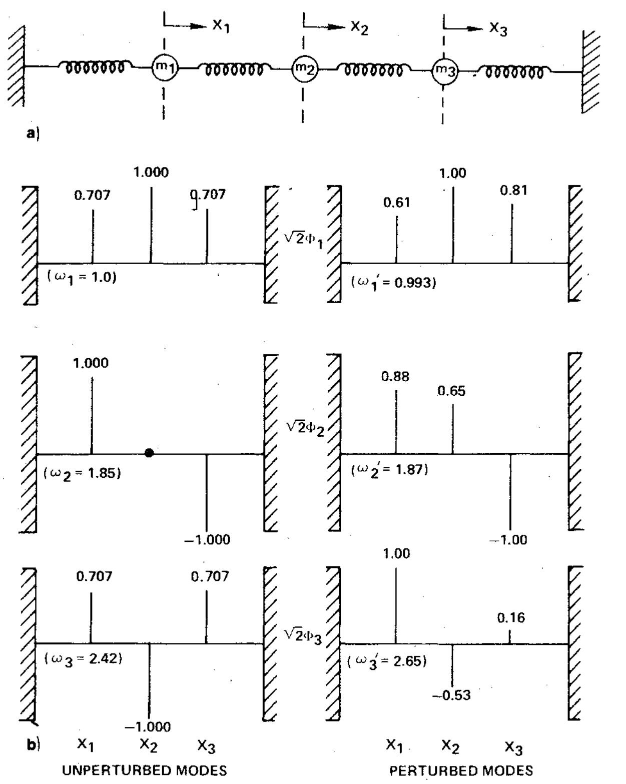

Fig. 1a) Linear spring mass system analyzed in first example. Masses m_1, m_2, m_3 are constrained to move in the horizontal direction as indicated by coordinates X_1, X_2, and X_3. b) Mode shapes and frequencies before (left-hand side) and after (right-hand side) perturbation in masses m_1, m_2, and m_3. Frequencies are all normalized to the unperturbed frequency of the first vibrational mode.

of motion. The modes are expressed as column vectors with components equal to the displacements of the first, second, and third masses, respectively

$$\Phi_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \Phi_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \Phi_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{42}$$

where the modes have been normalized according to the prescription

$$\Phi_n \Phi_n^T = 1 \quad \text{for } n = 1,2,3 \tag{43}$$

Suppose that we desire to decrease the amplitude of vibration of the first mass and increase that of the third with respect to that of the center mass when vibrating in the first mode. Also assume that the mode shapes and frequencies of the second and third modes are of no interest and can be left unconstrained. Let us determine the changes in the structure that are required in order to change the first mode to

$$\Phi_1' = \frac{1}{\sqrt{2}} \begin{pmatrix} 0.6 \\ 1.0 \\ 0.8 \end{pmatrix} \tag{44}$$

with no change in frequency $\Delta\omega_1 = 0$.

These changes in the first mode correspond to imposing three constraints, Δ_{11}, Δ_{12}, and Δ_{13}, and three structural changes are required which can be chosen from the seven available components (three masses and four springs). If the structural changes are restricted to changes in the three masses, the finite-element inversion equation [Eq. 33)] can be simplified to

$$\Delta_{nk} = -\frac{1}{2}\sum_{i=1}^{3}\Phi_{ni}\Phi_{ki}\left(\frac{\Delta m_i}{m}\right) \tag{45}$$

where use has been made of the normalization of the modes [Eq. (43)] and the equality of the unperturbed masses. If constraints Δ_{11}, Δ_{12}, and Δ_{13} are specified, Eq. (45) can be written in matrix form as

$$\begin{bmatrix}\Delta_{11}\\ \Delta_{12}\\ \Delta_{13}\end{bmatrix} = -\tfrac{1}{4}\begin{bmatrix}\tfrac{1}{2} & 1 & \tfrac{1}{2}\\ (\tfrac{1}{2})^{1/2} & 0 & (\tfrac{1}{2})^{1/2}\\ \tfrac{1}{2} & -1 & \tfrac{1}{2}\end{bmatrix}\begin{bmatrix}\Delta m_1/m\\ \Delta m_2/m\\ \Delta m_3/m\end{bmatrix} \tag{46}$$

where the mode shapes from Eq. (42) have been used to form the 3×3 matrix.

Equation (46) can be inverted to obtain the mass changes in terms of the constraints by taking the inverse of the 3×3 matrix. The result is

$$\begin{bmatrix}\Delta m_1/m\\ \Delta m_2/m\\ \Delta m_3/m\end{bmatrix} = -4\begin{bmatrix}\tfrac{1}{2} & (\tfrac{1}{2})^{1/2} & \tfrac{1}{2}\\ \tfrac{1}{2} & 0 & -\tfrac{1}{2}\\ \tfrac{1}{2} & (\tfrac{1}{2})^{1/2} & \tfrac{1}{2}\end{bmatrix}\begin{bmatrix}\Delta_{11}\\ \Delta_{12}\\ \Delta_{13}\end{bmatrix} \tag{47}$$

The constraints Δ_{11}, Δ_{12}, and Δ_{13} can be determined from Eq. (18) when $\Delta\omega_1 = 0$ gives $\Delta_{11} = 0$ directly. The admixture coefficients C_{12} and C_{13} are obtained by expressing the perturbed mode shapes from Eq. (44) as a linear combination of the unperturbed mode shapes. In matrix form

$$\begin{bmatrix}0.6\\ 1\\ 0.8\end{bmatrix} = \begin{bmatrix}(\tfrac{1}{2})^{1/2} & 1 & (\tfrac{1}{2})^{1/2}\\ 1 & 0 & -1\\ (\tfrac{1}{2})^{1/2} & -1 & (\tfrac{1}{2})^{1/2}\end{bmatrix}\begin{bmatrix}1\\ C_{12}\\ C_{13}\end{bmatrix} \tag{48}$$

Equation (48) has the solutions $C_{12} = -0.10$, $C_{13} = -0.01$. If these values, along with the values of ω_1, ω_2 and ω_3 from Eq. (41), are substituted into Eq. (18), the elements Δ_{12} and Δ_{13} can be calculated. The solution for the constraint vector is

$$\begin{bmatrix}\Delta_{11}\\ \Delta_{12}\\ \Delta_{13}\end{bmatrix} = \begin{bmatrix}0.000\\ 0.120\\ 0.024\end{bmatrix} \tag{49}$$

If Eq. (49) is substituted into Eq. (47) and the indicated matrix multiplication is carried out, the resulting fractional changes in mass are found to be

$$\Delta m_1/m = -0.39,\ \Delta m_2/m = 0.05,\ \Delta m_3/m = 0.29 \tag{50}$$

The validity and accuracy of the first-order solution of Eq. (50) can be checked by using the perturbed masses in Eq. (39) and determining the perturbed mode shapes and frequencies exactly. Equation (39) can be rewritten in the form

$$\begin{vmatrix}1-(m\omega'^2/2k)(1+\Delta m_1/m) & -\tfrac{1}{2} & 0\\ -\tfrac{1}{2} & 1-(m\omega'^2/2k)(1+\Delta m_2/m) & -\tfrac{1}{2}\\ 0 & -\tfrac{1}{2} & 1-(m\omega'^2/_k)(1+\Delta m_2/m)\end{vmatrix} = 0 \tag{51}$$

If the fractional mass changes from Eq. (50) are substituted into the determinant, a cubic equation for ω'^2 results. The roots are obtained by iteration and give the perturbed frequencies in terms of the unperturbed frequencies as

$$\omega'_1 = 0.993\omega_1;\ \Delta\omega_1/\omega_1 = -0.007 \tag{52a}$$

$$\omega'_2 = 1.01\omega_2;\ \Delta\omega_2/\omega_2 = 0.010 \tag{52b}$$

$$\omega'_3 = 1.10\omega_3;\ \Delta\omega_3/\omega_3 = 0.10 \tag{52c}$$

The frequency of the first mode indeed does remain constant to an accuracy of 0.7%, whereas the frequencies of the second and third modes, which have not been constrained, vary by 1% and 10%, respectively. The perturbed mode shapes can be determined by solving the set of linear equations that correspond to the secular determinant of Eq. (51) with the numerical values of the perturbed frequencies given by Eq. (52). The results are

$$\Phi'_1 = (\tfrac{1}{2})^{1/2}\begin{bmatrix}0.61\\ 1.00\\ 0.81\end{bmatrix};\ \Phi'_2 = (\tfrac{1}{2})^{1/2}\begin{bmatrix}0.88\\ 0.65\\ -1\end{bmatrix};\ \Phi'_3 = (\tfrac{1}{2})^{1/2}\begin{bmatrix}1\\ -0.53\\ 0.16\end{bmatrix} \tag{53}$$

The perturbed mode shapes are compared with the original mode shapes in Fig. 1b. As expected, the calculated mode shape for the perturbed first mode is in excellent agreement (better than 2%) with the constraint that was imposed on the fractional mass changes through Eq. (44). The shapes of the second and third modes change considerably as a result of the perturbation. This is not surprising, since no attempt was made to constrain the shapes of these modes. The shapes of the second and third modes also could be constrained if changes were made in the spring constants of the four springs. This would allow for a total of seven constraints to be imposed corresponding to all of the six admixture coefficients and one of the frequency changes, for example.

Continuous System Example: Cantilever Beam

The previous example illustrates the procedure involved in a simple inverse perturbation calculation and demonstrates the accuracy of the technique in the prediction of design changes. A more practical problem, that of modifying the modes of a uniform cantilever beam, is treated next, using the formal solution of Eq. (21) as the starting point.

The cantilever beam provides an interesting example because the solution for its modes is well known and can be expressed in closed form.[5] In addition, the cantilever beam provides a simple model for many important structures. The general solution for the free lateral vibration of a beam of uniform thickness h and uniform flexural rigidity (EI) is

$$\Phi_n(x) = K_{1n}\cosh\beta_n x + K_{2n}\sinh\beta_n x + K_{3n}\sin\beta_n x + K_{4n}\cos\beta_n x \tag{54}$$

with the following relation between the eigenvalue β_n and the mode frequency ω_n:

$$\beta_n^4 = m\omega_n{}^4/EI \tag{55}$$

where

β_n = eigenvalue of nth mode
ω_n = radian frequency of nth mode
m = mass per unit length

The mode shapes [Eq. (54)] and frequencies [Eq. (55)] can be substituted in Eq. (21), and the coefficients $B_{(nk)p}$ can be calculated for any choice of functions $f_p(x)$. In addition, the expression for the normalization integral for a beam with one free end will be used to simplify the expression for M_k, the modal mass. According to Ref. 6,

$$M_k = m \int_0^{\ell} \Phi_k^2(x)\,\mathrm{d}x = m\,\frac{\ell}{4}\,\Phi_k^2\Big|_{x=\ell} \tag{56}$$

for a beam with one free end at $x=\ell$. If the deflection at the free end is normalized to unity, the expression for $B_{(nk)p}$ of Eq. (21) can be simplified to

$$B_{(nk)p} = \frac{2}{\ell}\int_0^{\ell}\left[\frac{3\Phi_n''\Phi_k''}{\beta_n^4} - \Phi_n\Phi_k\right] f_p(x)\,\mathrm{d}x \tag{57}$$

Note that Eq. (56) was employed to simplify the calculation. In general, the mode shapes of Eq. (54) would be used to determine the modal mass directly, so that specific boundary conditions need not be applied once the mode shapes of the unperturbed design are known.

The choice of a set of trial functions $f_p(x)$ for the change in thickness Δh for this problem will be made to simplify the evaluation of the integrals in Eq. (57). The thickness changes will be implemented by dividing the length of the beam into N segments and allowing the thickness of each segment to be varied independently. The expression for $f_p(x)$ in this case is

$$f_p(x) = \begin{cases} 1 & \text{for } \ell_{p-1} < x < \ell_p \\ 0 & \text{elsewhere} \end{cases} \tag{58}$$

and the coefficients A_p of Eq. (19) are equal to the fractional change in thickness of the pth segment. If Eq. (58) is substituted into Eq. (57), the expression for $B_{(nk)p}$ simplifies further to

$$B_{(nk)p} = \frac{2}{\ell}\int_{\ell_{p-1}}^{\ell_p}\left[\frac{3\Phi_n''\Phi_k''}{\beta_n^4} - \Phi_n\Phi_k\right]\mathrm{d}x \tag{59}$$

Equation (59) for the inversion coefficients $B_{(nk)p}$ of the cantilever beam is evaluated explicitly in the Appendix. Consider an ideal case in which the fixed end of the beam is perfectly clamped. The solutions for the coefficients and the eigenvalues to be used in Eq. (54) are well known for this case, as discussed in Ref. 4. The numerical results for the first three modes are summarized in Table 1.

Now consider the following practical design problem. The tip deflection of the second vibrational mode is to be reduced by approximately 30% relative to the amplitude of its antinode. The frequency of the second vibrational mode is to be held constant. The shape of the second vibrational mode for a cantilever beam of uniform thickness is shown in Fig. 2, with the tip deflection normalized to unity. The ratio of the tip deflection to the deflection at the antinode for the second vibrational mode is 1.00/0.72 = 1.39/1, as shown in Fig. 2. Also shown in Fig. 2 is the normalized mode shape obtained by adding 20% of the third vibrational mode to the second. The ratio of the tip deflection to the deflection of the first antinode for the latter mode shape is 1/1, a decrease of 28% when compared with the unperturbed second vibrational mode. The desired design change thus can be implemented in an approximate fashion by imposing a pair of constraints corresponding to a specification of the admixture coefficient C_{23} and the frequency change $\Delta\omega_2$, as follows:

$$C_{23} = 0.20, \quad \Delta\omega_2 = 0.00 \tag{60}$$

In addition, the admixture coefficient C_{21}, which specifies the amount of the first mode shape that is added to the second, is set equal to zero. These three constraints can be satisfied in a straightforward way by dividing the beam into three segments and making independent changes in the thickness of the three segments. However, the three segments should be of unequal length if the admixture coefficients C_{21} and C_{23} are to be specified independently. This is, if $\ell_p = \ell/3$ in Eq. (59), the two rows $B_{(23)p}$ and $B_{(21)p}$ are proportional, and the matrix is

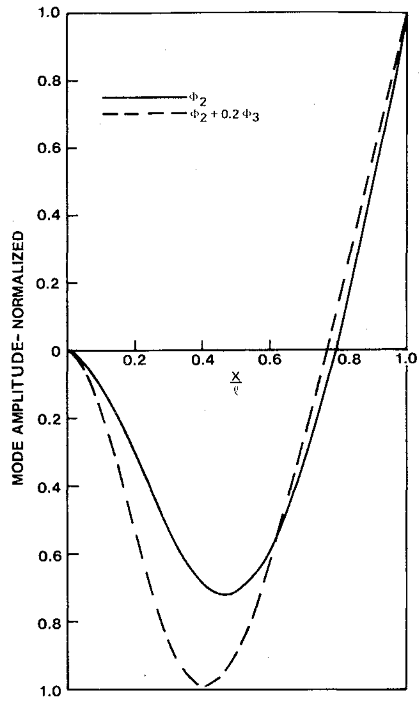

Fig. 2 Normalized mode shapes for a clamped cantilever beam. Solid curve is the mode for the second vibrational mode. Dashed curve is an altered mode shape obtained by adding 20% of the third vibrational mode to the second. Both curves are normalized such that the amplitude of the tip deflection is unity.

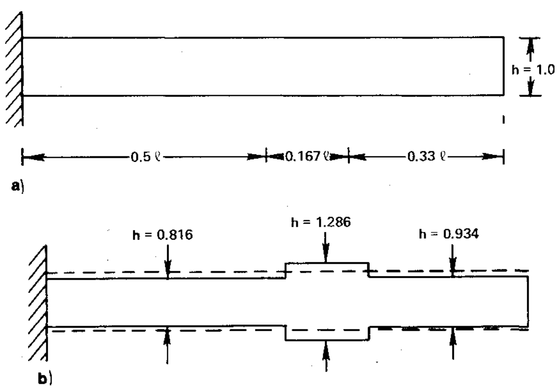

Fig. 3a) Cantilever beam segments selected for thickness perturbation. Segments of unequal length are chosen to insure independence of constraints. b) Thickness changes necessary to implement change in mode shape shown in Fig. 2, as calculated by inverse perturbation algorithm.

Table 1 Solutions for free lateral vibration of cantilever beam – fixed end clamped

Mode	$\beta_n\ell$	K_{1n}	K_{2n}	K_{3n}	K_{4n}
1	1.875	0.50004	−0.36708	0.36708	−0.50004
2	4.694	0.50005	−0.50928	0.50928	−0.50005
3	7.855	0.49988	−0.49949	0.49949	−0.49988

singular, an indication that the constraints Δ_{21} and Δ_{23} are not independent. To overcome this limitation, the beam is divided into three unequal segments of lengths ($\ell_1=\ell/2$, $\ell_2=\ell/6$, $\ell_3=\ell/3$), as shown in Fig. 3a. The required fractional changes in the thickness of the three segments (A_1,A_2 and A_3) are related to the constraints Δ_{21}, Δ_{22}, and Δ_{23} by Eq. (20) with $N=3$. In matrix form, Eq. (20) can be written as

$$\begin{bmatrix} B_{(21)1} & B_{(21)2} & B_{(21)3} \\ B_{(22)1} & B_{(22)2} & B_{(22)3} \\ B_{(23)1} & B_{(23)2} & B_{(23)3} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \Delta_{21} \\ \Delta_{22} \\ \Delta_{23} \end{bmatrix} \tag{61}$$

The numerical values from Table 1 are used in Eqs. (A3-A7) of the Appendix to determine the elements of the inversion matrix $B_{(nk)p}$. The elements of the constraint vector Δ_{nk} are determined by substituting numerical values for C_{nk}, $\Delta\omega_n/\omega_n$, β_k into Eq. (18). This results in the following matrix equation for A_1, A_2 and A_3:

$$\begin{bmatrix} 0.0402 & 0.0082 & -0.04839 \\ 0.5294 & 0.2958 & 0.1747 \\ 2.373 & -0.9747 & -1.398 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ -0.684 \end{bmatrix} \tag{62}$$

where $C_{21}=0$, $\Delta\omega_{=}=0$, and $C_{23}=0.20$. The solutions of Eq. (62) are $A_1=-0.1838$, $A_2=0.2856$, and $A_3=-0.0660$. The modified beam thickness profile that results is shown in Fig. 3b. The results are self-consistent in that the fractional changes in thickness are the same order of magnitude as the admixture coefficient C_{23}.

Conclusions

The two examples were included to demonstrate the inverse perturbation method using structures for which the modes are known in simple analytical form. It is expected that the method will be used to the greatest advantage in the iterative design of complicated structures for which the unperturbed modes are known in numerical form, either from a finite-element calculation or from measurements on an existing prototype. These applications will require the development of iterative computer codes based on the inverse perturbation algorithms discussed in this paper. Of equal importance at this time is a determination of the accuracy of the inverse perturbation calculations when used as a design tool. This question was addressed, somewhat, in the first of the illustrative examples. However, a more general treatment of the accuracy of the inverse perturbation method is required for the treatment of more complex design problems. Closely related to the question of accuracy is that of the conditioning of the matrices to be inverted. The latter would involve development of consistent rules for choosing the trial functions for the independent structural changes for a desired set of changes in mode shape and frequency.

Appendix: Evaluation of Inversion Coefficients for Cantilever Beam

The expressions for the mode shapes [Eq. (54)] will be written in the following concise form for purposes of computation:

$$\Phi_n(x)=g_n(x)+h_n(x) \tag{A1}$$

$$\Phi_n''(x)=\beta_n^2[g_n(x)-h_n(x)] \tag{A2}$$

where

$$g_n(x)=K_{1n}\cosh\beta nx+K_{2n}\sinh\beta nx \tag{A3}$$

$$h_n(x)=K_{3n}\sin\beta nx+K_{4n}\cos\beta nx \tag{A4}$$

Equations (A1) and (A3) are substituted into Eq. (59), and integration by parts is employed twice to obtain the following results:

$$\begin{aligned} B_{(nk)p} &= \frac{2}{\ell}\int_{\ell p-1}^{\ell p} \mathrm{d}x\left[\left(\frac{3\beta_k^2}{\beta_n^2}-1\right)(g_n g_k+h_n h_k)-\left(\frac{3\beta_k^2}{\beta_n^2}+1\right)(g_n h_k+g_k h_n)\right] \\ &= \frac{2}{\ell}\left\{\left(\frac{3\beta_k^2}{\beta_n^2}-1\right)[A_{nk}(\ell_p)-A_{nk}(\ell_{p-1})]-\left(\frac{3\beta_k^2}{\beta_n^2}+1\right)[D_{nk}(\ell_p)-D_{nk}(\ell_{p-1})]\right\} \end{aligned} \tag{A5}$$

where

$$A_{nk}(x)=\begin{cases} \dfrac{g_n'(x)g_k(x)-g_n(x)g_k'(x)+h_n(x)h_k'(x)-h_n'(x)h_k(x)}{\beta_n^2-\beta_k^2}, & n\neq k \\[2ex] \dfrac{g_n(x)g_n'(x)-h_n(x)h_n'(x)}{2\beta_n^2}+\dfrac{x}{2}[K_{1n}^2-K_{2n}^2+K_{3n}^2+K_{4n}^2], & n=k \end{cases} \tag{A6}$$

$$D_{nk}(x)=\frac{g_n'(x)h_k(x)-g_n(x)h_k'(x)-h_n'(x)g_k(x)+h_n(x)g_k'(x)}{\beta_n^2+\beta_k^2} \tag{A7}$$

Note that $A_{nk}=A_{kn}$ and $D_{nk}=D_{kn}$, so that both $B_{(nk)p}$ and $B_{(kn)p}$ can be calculated if A_{nk} and D_{nk} are known. The functions g_n, h_n, g_k, h_k, and their first derivatives can be evaluated using Eqs. (A3) and (A4) with numerical values for the eigenvalues and coefficients. The numerical values of the functions and their first derivatives then are substituted into Eqs. (A6) and (A7) to evaluate the A_{nk} and D_{nk}. The latter then are used to calculate $B_{(nk)p}$ and $B_{(kn)p}$ using Eq. (A5).

References

[1]Stetson, K.A., "A Perturbation Method of Structural Design Relevant to Holographic Vibration Analysis," *AIAA Journal*, Vol. 13, April 1975, pp. 457-459.

[2]Berman, A. and Flannelly, W.G., "Theory of Incomplete Models of Dynamic Structures," *AIAA Journal*, Vol. 9, Aug. 1971, pp. 1481-1487.

[3]Chen, J.C. and Wada, B.K., "Criteria for Analysis-Test Correlation of Structural Dynamic Systems," *Journal of Applied Mechanics*, Vol. 42, June 1975, pp. 471-477.

[4]Huebner, K.H., *The Finite Element Method for Engineers*, Wiley, New York, 1975.

[5]Hurty, W.C. and Rubenstein, M.F., *Dynamics of Structures*, Prentice Hall, Englewood Cliffs, N.J., 1964.

[6]Timoshenko, S., *Vibration Problems in Engineering*, D. Van Mortravel Co., Princeton, N.J. 1955.